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LINEAR AND QUADRATIC INTEGRALS OF A COMPOUND MECHANICAL SYSTEM†

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A mechanical system consisting of an unchangeable subsystem (the carrier-body) and a changeable part (the working body) is considered. The motion of particles of the working body with respect to the carrier is programmed and specified. The structurally changeable part of the system has a configuration that varies with time and may be either constant or variable in its composition. The system is situated in a uniform gravitational field. The motion of the system is considered in the case when its centre of inertia is not a fixed point of the carrier. Criteria are formulated for the existence of linear and quadratic integrals, and the explicit form of the integrals is determined. A mechanical interpretation is given for the integrals and the conditions for their existence.

A number of publications‡ have been devoted to a system of changeable configuration, acted upon by reactive forces produced when a working body whose particles have given absolute velocities is detached, while the carrier-body has a fixed point. In particular, conditions have been established for the existence of a linear integral and a quadratic integral. Equations of motion exist for the case in which the velocities with which the particles of the working body detach from the carrier-body are given.§

The mathematical model considered below includes Aminov's and Makeyev's models (models A and M). For model A we solve the problem of finding integrals, and for model M the integrals are written out explicitly, and the number of criterial conditions is reduced compared with previous treatments.

1. Let E_i be an inertial orthobasis, let E_2 be a basis rigidly attached to a carrier and let E_3 be the principal orthobasis, given by the principal axes of inertia of the system at a fixed point O of the carrier; $\mathbf{x}_{ij}$ is the instantaneous angular velocity of basis E_i relative to E_j. If the system is considered as a collection of material points M_n, its angular momentum $\mathbf{G}$ relative to O in motion relative to the basis E_i may be written as follows ($\mathbf{r}_n = \mathbf{OM}_n$):

$$\mathbf{G}_i = \sum m_n \mathbf{r}_n \times (\mathbf{r}_n)^{\cdot}_{E_i} \tag{1.1}$$

Changing to the derivative in the basis E_j, we obtain

$$\mathbf{G}_i = \mathbf{G}_j + J\mathbf{x}_{ji} \tag{1.2}$$

where J is the inertia operator of the system at O, A_k and $\mathbf{e}_k$ denote the eigenvalues and corresponding eigenvectors of the inertia operator.

The absolute angular momentum of the system relative to O is denoted by $\mathbf{G}$, $\mathbf{G} = \mathbf{G}_1$.

The equations of motion [1], written for the angular velocity $\mathbf{x}_{21}$, may be expressed as

$$\dot{\mathbf{G}} = \mathbf{G} \times \mathbf{x}_{21} + \Lambda \mathbf{x}_{21} + \mathbf{s} \times \mathbf{a} + \mathbf{L}, \quad \dot{\mathbf{s}} = \mathbf{s} \times \mathbf{x}_{21} \tag{1.3}$$

where the dot denotes differentiation with respect to time in E_2, $\mathbf{s}$ is the unit vector in the vertical direction, $\mathbf{r}_c = \mathbf{OC}$, $\mathbf{a} = P\mathbf{r}_c$ and P is the weight of the body.

The operator Λ and the function $\mathbf{L}(t)$ are given. For model A

$$\mathbf{L}(t) = \mathbf{L}^r(t) + \sum \dot{m}_n \mathbf{r}_n \times \mathbf{v}_n^r, \quad \Lambda \mathbf{x} = \sum \dot{m}_n \mathbf{r}_n \times (\mathbf{x} \times \mathbf{r}_n) \tag{1.4}$$

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‡MAKEYEV N. N., Integral manifolds of the equations of dynamics of compound mechanical systems. Doctorate dissertation, St Petersburg, 1992.

§MAKEYEV N. N., Linear and quadratic integrals of a compound system. Unpublished paper. Dep. No. 1657–B89 at VINITI 14.03.89. Saratov, 1989.

where $\mathbf{L}^r$ is the principal moment of the reactive forces about the point O and $\mathbf{v}_n^r = \dot{\mathbf{r}}_n$ is the velocity of a particle relative to the carrier.

The equations of motion for model M may also be written in the form of (1.3), with $\Lambda \equiv 0$ and $\mathbf{L}(t)$ the principal moment of the quasi-reactive forces, which is given for this model.

To construct the linear integral, it is convenient to use a system equivalent to (1.3)

$$J(\mathbf{x})^{\cdot}_{E_3} = (J\mathbf{x})\times\mathbf{x} + Z\mathbf{x} + \mathbf{s}\times\mathbf{a} + \mathbf{N}, \quad (\mathbf{s})^{\cdot}_{E_3} = \mathbf{s}\times(\mathbf{x}+\mathbf{b}) \tag{1.5}$$

For model A: $\mathbf{x} = \mathbf{x}_{21}$, $\mathbf{b} = \mathbf{x}_{32}$, $Z = \Lambda - (J)^{\cdot}_{E_3} + M(\mathbf{G}_2) - M(\mathbf{x}_{32})J$, $\mathbf{N} = \mathbf{L} - (G_2)^{\cdot}_{E_3} - \mathbf{x}_{32} \times \mathbf{G}_2$.

For model M: $\mathbf{x} = \mathbf{x}_{31}$, $\mathbf{b} = 0$, $Z = M(\mathbf{G}_3) - (J)^{\cdot}_{E_3}$, $\mathbf{N} = \mathbf{L} - (\mathbf{G}_3)^{\cdot}_{E_3}$.

Here and below $M(\mathbf{c})$ is the vector multiplication operator, $M(\mathbf{c})\mathbf{x} = \mathbf{c} \times \mathbf{x}$. When $\mathbf{N} = 0$, $\mathbf{b} = 0$, $Z = 0$, system (1.5) is the Euler–Poisson system. When $\mathbf{b} = 0$, $\mathbf{N} = -(\mathbf{d})^{\cdot}_{E_3}$, $Z = M(\mathbf{d})$, it describes the motion of a gyrostat with gyrostatic moment $\mathbf{d}$.

We shall assume throughout that $\Lambda(t)$ and $\mathbf{L}(t)$ are continuous and $J(t)$, $\mathbf{G}(t)$, $\mathbf{a}(t)$ are continuously differentiable functions in the interval $[0, +\infty]$.

2. Conditions for the existence of an integral linear in $\mathbf{x}$, $F(\mathbf{x}, t) = \mathbf{m} \cdot \mathbf{x} + \varphi(t)$, are obtained by differentiating $F(\mathbf{x}, t)$ along trajectories of system (1.5). One of the necessary conditions is symmetry of the inertia ellipsoid. If such symmetry exists (e.g. $A_1 = A_2 \neq A_3$), the necessary and sufficient conditions for the existence of the integral are

$$m^{(1)} = m^{(2)} = 0, \quad a^{(1)} = a^{(2)} = 0, \quad z_{31} = z_{32} = 0 \tag{2.1}$$

and the integral itself in the form $m^{(3)}x^{(3)} + \varphi = \text{const}$ is equivalent to the definition of $x^{(3)}$ as a solution of the equation $A_3\dot{x}^{(3)} = z_{33}x^{(3)} + N^{(3)}$. Throughout, $c^{(k)}$ will denote the projection of the vector $\mathbf{c}$ onto an axis of the basis E_3, $c^{(k)} = \mathbf{c} \cdot \mathbf{e}_k$. Obviously, if conditions (2.1) are satisfied, the last equation may be obtained directly from (1.5). In the classical case of a symmetric gyroscope this integral yields the well-known condition $\mathbf{x}^{(3)} = \text{const}$, where z_{ij} are the matrix elements of the operator Z in the basis E_3.

On the assumption that all the A_i are distinct, we shall derive conditions for the existence of a linear integral invariant (or particular integral) $\Phi(\mathbf{x}, \mathbf{s}, t) = 0$. Without loss of generality, it will suffice to consider the cases $\Phi = \mathbf{n} \cdot \mathbf{x} - 1$ and $\Phi = \mathbf{n} \cdot \mathbf{x}$.

We shall find conditions for the existence of a linear particular integral

$$\mathbf{n} \cdot \mathbf{x} = 1 \tag{2.2}$$

Differentiating (2.2) along trajectories of system (1.5), we obtain the condition

$$(\mathbf{n})^{\cdot}_{E_3} \cdot \mathbf{x} + \mathbf{n} \cdot J^{-1}((J\mathbf{x})\times\mathbf{x} + Z\mathbf{x} + \mathbf{s}\times\mathbf{a} + \mathbf{N}) = 0 \tag{2.3}$$

which must hold for all t, $\mathbf{s}$ and for all $\mathbf{x}$ satisfying the constraint (2.2). One of the necessary conditions is $\mathbf{r}_c \parallel J^{-1}\mathbf{n}$. By (2.2), we can put $\mathbf{x} = \mathbf{n}\,|\mathbf{n}|^{-2} + \mathbf{l}$, where $\mathbf{l} \cdot \mathbf{n} = 0$. Condition (2.3) is equivalent to the requirement that the terms of order up to two inclusive vanish with respect to $\mathbf{l}$, i.e.

$$\mathbf{n} \cdot ((\mathbf{n})^{\cdot}_{E_3} + J^{-1}Z\mathbf{n} + J^{-1}\mathbf{N}|\mathbf{n}|^{-2} + J^{-1}((J\mathbf{n})\times\mathbf{n})|\mathbf{n}|^{-2}) = 0 \tag{2.4}$$

$$\mathbf{l} \cdot (\mathbf{n})^{\cdot}_{E_3} + \mathbf{n} \cdot J^{-1}(Z\mathbf{l} + (J\mathbf{n}\times\mathbf{l} + J\mathbf{l}\times\mathbf{n})|\mathbf{n}|^{-2}) = 0 \quad \forall\mathbf{l}:\ \mathbf{n}\cdot\mathbf{l} = 0 \tag{2.5}$$

$$\mathbf{n} \cdot J^{-1}((J\mathbf{l}) \times \mathbf{l}) = 0 \quad \forall\mathbf{l}:\ \mathbf{n} \cdot \mathbf{l} = 0 \tag{2.6}$$

As shown by an analysis of condition (2.6), this condition will only hold when

$$n^{(2)} = 0, \quad \alpha_1(n^{(1)})^2 = \alpha_3(n^{(3)})^2 \tag{2.7}$$

Here and below, we have assumed that $A_1 \geqslant A_2 \geqslant A_3$, $\alpha_k = (A_i - A_j)A_k^{-1}\delta_{ijk}$, where $\delta_{ijk} = 1$ if (i, j, k) is a cyclic permutation of (1, 2, 3) and $\delta_{ijk} = -1$ otherwise. It follows from (2.7) that the vector $\mathbf{n}$ lies in a plane orthogonal to the middle axis of the inertia ellipsoid and may be inclined to the minor axis at angles $\pm\beta$, where $\operatorname{tg}\beta = \sqrt{(\alpha_1\alpha_3^{-1})}$.

Condition (2.5) is equivalent to the collinearity condition

$$(\mathbf{n})^{\cdot}_{E_3} + Z^T J^{-1}\mathbf{n} + \mathbf{c}|\mathbf{n}|^{-2} = q\mathbf{n} \tag{2.8}$$

where $\mathbf{c} = J(\mathbf{n} \times J^{-1}\mathbf{n}) - (J\mathbf{n}) \times (J^{-1}\mathbf{n})$, $\mathbf{n} \cdot \mathbf{c} = 0$. Taking the scalar product of (2.8) and $\mathbf{n}$, we find that $q = \mathbf{n} \cdot ((\mathbf{n})^{\cdot}_{E_3} + Z^T J^{-1}\mathbf{n})|\mathbf{n}|^{-2}$. It follows from (2.7) that $\mathbf{n} \cdot J^{-1}((J\mathbf{n}) \times \mathbf{n}) = 0$ and (2.4) takes the form $\mathbf{n} \cdot ((\mathbf{n})^{\cdot}_{E_3} + J^{-1}Z\mathbf{n} + J^{-1}\mathbf{N}|\mathbf{n}|^2) = 0$, which enables us to find the parameter q, $q = -\mathbf{n} \cdot J^{-1}\mathbf{N}$.

For this value of q condition (2.8) yields

$$(\mathbf{n})^{\cdot}_{E_3} + Z^T J^{-1}\mathbf{n} + \mathbf{c}|\mathbf{n}|^{-2} = -(\mathbf{n} \cdot J^{-1}\mathbf{N})\mathbf{n} \tag{2.9}$$

which is equivalent to the system of conditions (2.4) and (2.5). For the two possible directions of $\mathbf{n}$ given by condition (2.7), the vector $\mathbf{c}$ is collinear with the middle axis of the ellipsoid, $\mathbf{c} = \pm|\mathbf{n}|^2\sqrt{(\alpha_1\alpha_3)}\mathbf{e}_2$.

If the inertia ellipsoid at O is not a sphere, this result may be expressed as follows. The linear integral (2.2) has the form

$$n^{(1)}x^{(1)} + n^{(3)}x^{(3)} = 1 \tag{2.10}$$

subject to satisfaction of the necessary and sufficient conditions

$$\begin{gathered} (\mathbf{n})^{\cdot}_{E_3} + Z^T J^{-1}\mathbf{n} + (\mathbf{L} \cdot J^{-1}\mathbf{n})\mathbf{n} = \mp\sqrt{\alpha_1\alpha_3}\,\mathbf{e}_2 \\ \mathbf{r}_c \| J^{-1}\mathbf{n}, \quad n^{(2)} = 0, \quad \sqrt{\alpha_1}\,n^{(1)} = \pm\sqrt{\alpha_3}\,n^{(3)} \end{gathered} \tag{2.11}$$

The initial conditions must satisfy condition (2.10); the complete number of scalar conditions here is at most eight.

The linear relationship obtained by projecting the differential condition (2.11) onto $\mathbf{e}_2$, together with condition (2.7), yields

$$\begin{gathered} n^{(1)} = \cos\gamma(x^*)^{-1}, \quad n^{(3)} = \sin\gamma(x^*)^{-1} \\ \gamma = \pm\beta, \quad \operatorname{tg}\beta = \sqrt{\alpha_1\alpha_3^{-1}}, \quad x^* = -\zeta_{12}\sin\gamma\alpha_1^{-1} - \zeta_{32}\cos\gamma\alpha_3^{-1}, \quad z_{ij} = A_i\zeta_{ij} \end{gathered} \tag{2.12}$$

The integral (2.10) may now be written explicitly as

$$x^{(1)}\cos\gamma + x^{(3)}\sin\gamma = x^* \tag{2.13}$$

The two differential conditions for $n^{(1)}$, $n^{(3)}$ obtained from (2.11) may be written, using (2.12), as

$$\dot{x}^* - x^*\left(\zeta_{11}\cos^2\gamma + \zeta_{33}\sin^2\gamma + (\zeta_{13} + \zeta_{31})\frac{\sin 2\gamma}{2}\right) = \frac{N^{(1)}\cos\gamma}{A_1} + \frac{N^{(3)}\sin\gamma}{A_3} \tag{2.14}$$

$$\dot{f} + \zeta_{13} + (\zeta_{33} - \zeta_{11})f - \zeta_{31}f^2 = 0, \quad f = \operatorname{tg}\gamma \tag{2.15}$$

The condition $\mathbf{r}_c \| J^{-1}\mathbf{n}$, considered in the case $\sqrt{(\alpha_1)}n^{(1)} = \pm\sqrt{(\alpha_3)}n^{(3)}$, is of the same form as Hess's well-known configuration conditions for a rigid body

$$\sqrt{A_1(A_2 - A_3)}\,r_c^{(1)} \mp \sqrt{A_3(A_1 - A_2)}\,r_c^{(3)} = 0, \quad r_c^{(2)} = 0 \tag{2.16}$$

Denoting the angle between $\mathbf{r}_c$ and the semi-minor axis of the inertia ellipsoid by φ, we obtain for condition (2.16)

$$A_1\sqrt{\alpha_1}\cos\varphi = \pm A_3\sqrt{\alpha_3}\sin\varphi \tag{2.17}$$

and the integral (2.13) becomes

$$A_1x^{(1)}\cos\varphi + A_3x^{(3)}\sin\varphi = x^*\sqrt{1 + \alpha_1\alpha_3 A_1 A_3 A_2^{-1}} \tag{2.18}$$

This integral expresses the projection of the angular momentum onto the direction of $\mathbf{r}_c$.

We have thus shown that the linear integral (2.2) has the form of (2.13). Necessary and sufficient conditions for its existence are the configuration conditions (2.16) and the differential relations (2.14) and (2.15).

Repeating the above reasoning for the integral $\mathbf{n}\cdot\mathbf{x} = 0$, it can be shown that its form and the conditions for its existence are obtained from (2.13)–(2.16) by setting $x^* = 0$.

If the inertia ellipsoid has an axis of rotation, for example if $A_1 = A_2 \neq A_3$, the necessary conditions for the existence of the particular integral (2.2) are still (2.1), and so in this case the integral $F = \mathbf{m}\cdot\mathbf{x} + \varphi(t)$ analysed above exists.

Cases in which $\mathbf{N}$ is regulated under conditions that ensure that $\mathbf{N}$ is orthogonal to or collinear with $\mathbf{x}$ may be analysed by letting $\mathbf{N}$ have the form $\mathbf{N} = \mathbf{u}(t)\times\mathbf{x}$ or $\mathbf{N} = p(t)\mathbf{x}$, which preserves the form (1.5) of the dynamical system, incorporating $\mathbf{N}$ in the linear term $Z\mathbf{x}$, but adding to the operator Z in the first case $M(\mathbf{u})$ and in the second pE (where E is the identity operator). If conditions (2.14) and (2.15) are satisfied, one must set $N^{(1)} = N^{(3)} = 0$ and vary ζ_{ij}. The configuration condition (2.16) is retained.

The presence of the integral (2.13) enables one to reduce the order of system (1.5). If integrals of type (2.13) for $\gamma = \beta$ and $\gamma = -\beta$ exist, the components $\mathbf{x}$ are uniquely defined, since (for different A_i) condition (2.16) may be satisfied only if $r_c = 0$ and $x^{(1)} = -\zeta_{32}\alpha_3^{-1}$, $x^{(3)} = -\zeta_{12}\alpha_1^{-1}$, whatever the sign. We now derive from (1.5) a linear equation for $x^{(2)}$

$$\dot{x}^{(2)} - \zeta_{22}x^{(2)} = (\alpha_2\zeta_{12}\zeta_{32} - \alpha_1\zeta_{21}\zeta_{32} - \alpha_3\zeta_{23}\zeta_{12})(\alpha_1\alpha_3)^{-1} + N^{(2)}A_2^{-1}$$

3. Let us consider the application of our general results for system (1.5) to the analysis of model M. In that case

$$z_{ii} = -\dot{A}_i,\quad z_{12} = -z_{21} = -G_3^{(3)},\quad z_{13} = -z_{31} = G_3^{(2)},\quad z_{23} = -z_{32} = -G_3^{(1)}$$

Using condition (2.17) with the plus sign, we can rewrite the necessary condition (2.15) for the existence of a linear integral as

$$\dot{\varphi} = -G_3^{(2)}A_2^{-1} \tag{3.1}$$

Put $g = x^*\nu^{-1}$, $\nu = \cos\gamma(A_1\cos\varphi)^{-1}$. Then $g = (G_3{}^{(1)}\sin\varphi - G^{(3)}{}_3\cos\varphi)(\alpha_1\alpha_3)^{-1/2}$. Condition (2.14) may be written, in view of (3.1), as a condition for g

$$\dot{g} - (\alpha_1\alpha_3)^{1/2}A_2^{-1}G_3^{(2)}g = N_{rc} \tag{3.2}$$

where $N_{rc} = N_1\cos\varphi + N_3\sin\varphi$ is the projection of the vector $\mathbf{N}$ onto the direction $\mathbf{r}_c$. The integral (2.13) becomes

$$G^{(1)}\cos\varphi + G^{(3)}\sin\varphi = A_2(G_3^{(3)}\sin\varphi(A_2 - A_3)^{-1} - G_3^{(1)}\sin\varphi(A_1 - A_2)^{-1}) \tag{3.3}$$

Using Hess's condition, we deduce from (3.2) that

$$(A_2(G_3^{(3)}\sin\varphi(A_2 - A_3)^{-1} - G_3^{(1)}\cos\varphi(A_1 - A_2)^{-1})^{\cdot} = L_{rc} \tag{3.4}$$

Thus, if conditions (2.17), (3.1) and (3.4) hold, a linear particular integral exists and it has the form of (3.3).

4. We will now clarify the mechanical meaning of the linear integral and the conditions for its existence. Let E_5 be an orthogonal basis obtained from E_3 by rotating it through the angle φ about the middle axis of the inertia ellipsoid. Let $\mathbf{e}_{5i}$ be the unit vectors of E_5. The coordinates of $\mathbf{e}_{5i}$ in E_3 are $\mathbf{e}_{51} = (\cos\varphi; 0; \sin\varphi)$, $\mathbf{e}_{52} = (0; 1; 0)$, $\mathbf{e}_{53} = (-\sin\varphi; 0; \cos\varphi)$. It can be shown that (here $G^{(5i)} = \mathbf{G}\cdot\mathbf{e}_{5i}$)

$$A_2(G_3^{(3)}\sin\varphi(A_2 - A_3)^{-1} - G_3^{(1)}\cos\varphi(A_1 - A_2)^{-1}) = G_3^{(51)} + G_3^{(53)}(\alpha_1\alpha_3)^{1/2} \tag{4.1}$$

In view of (4.1), the integral (3.3) yields an expression for the projection of the angular momentum (AM) of the system onto the direction $\mathbf{r}_c$ in terms of the projection onto $\mathbf{r}_c$ and the orthogonal direction $\mathbf{e}_{53}$ of the AM of the system in its motion relative to the principal basis

$$G_{rc} = G_3^{(51)} + G_3^{(53)}(\alpha_1\alpha_3)^{-1/2} \tag{4.2}$$

Since $\mathbf{G} - \mathbf{G}_3 = J\mathbf{x}_{31} = \mathbf{G}^e$ is the AM of the system in translational motion together with the principal basis E_3, the integral (4.2) may be reduced to the form

$$G_{rc}^e = G_3^{(53)}(\alpha_1\alpha_3)^{-1/2} \tag{4.3}$$

which expresses the projection onto $\mathbf{r}_c$ of the AM of the system in translational motion with basis E_3 in terms of the projection of the AM of the system in its motion relative to the basis E_3 onto the direction orthogonal to $\mathbf{r}_c$ and the middle axis of the inertia ellipsoid. If $G_3^{(53)} = 0$, one obtains an analogue of the linear integral for a rigid body: $G_{rc}^e = 0$.

The necessary condition (3.4) may be written in the form

$$(G_3^{(51)} + G_3^{(53)}(\alpha_1\alpha_3)^{-1/2})^{\cdot} = L_{rc} \tag{4.4}$$

If Hess's condition holds, the rotation of $\mathbf{r}_c$ takes place in a plane orthogonal to the middle axis of the inertia ellipsoid, and the vector $J\mathbf{x}_{53}$ equals the AM of the system in translational motion: rotation together with $\mathbf{r}_c$ relative to E_3. Since in E_3 we have $\mathbf{x}_{53} = (0; -\dot{\varphi}; 0)$, the necessary condition for the existence of the integral (3.1) expresses the fact that the projection of the AM of the system moving relative to E_3 onto the middle axis of the inertia ellipsoid is equal to the projection onto the same axis of the AM of the system rotating with $\mathbf{r}_c$ relative to E_3.

Comparing the integral (3.3) with condition (3.4), we get

$$\dot{G}_{rc} = N_{rc} \tag{4.5}$$

Consequently, in this motion the derivative of the projection of the AM of the system onto the direction of the barycentric vector equals the projection onto the same direction of the principal moment of the quasi-reactive forces.

We will now compare property (4.5) with the theorem on the variation of AM: $(\mathbf{G})^{\cdot}_{E_5} + \mathbf{x}_{51} \times \mathbf{G} = \mathbf{N} + \mathbf{s} \times \mathbf{a}$. Hence, bearing in mind that the direction of $\mathbf{r}_c$ is fixed in E_5, we obtain $\mathbf{r}_c^0 \cdot (\mathbf{x}_{51} \times \mathbf{G}) + \dot{G}_{rc} = N_{rc}$, which agrees with (4.5) since it can be shown that when the linear integral exists the vectors $\mathbf{r}_c^0$, $\mathbf{x}_{51}$, $\mathbf{G}$ are collinear.

Taking (3.3) and Hess's condition into consideration, we see that the following identity holds in E_3

$$\mathbf{G} - G_{rc}\mathbf{r}_c^0 = (G^{(1)}\sin^2\varphi - G^{(3)}\sin\varphi\cos\varphi;\ G^{(2)};\ G^{(3)}\cos^2\varphi - G^{(1)}\cos\varphi\sin\varphi) \tag{4.6}$$

Define a vector $\mathbf{x}_c^a = \mathbf{r}_c^0 \times (\mathbf{r}_c^0)^{\cdot}_{E_1} = \mathbf{r}_c^0 \times ((\mathbf{r}_c^0)^{\cdot}_{E_3} + \mathbf{x}_{31} \times \mathbf{r}_c^0)$. Expressing $\mathbf{x}_{31}$ in terms of G, we see that in E_3

$$\mathbf{x}_c^a = A_2^{-1}(G^{(1)}\sin^2\varphi - G^{(3)}\sin\varphi\cos\varphi;\ G^{(2)} - G^{(3)} - A_2\dot{\varphi}; G^{(3)}\cos^2\varphi - G^{(1)}\cos\varphi\sin\varphi)$$

Comparing $\mathbf{x}_c^a$ with (4.6) and assuming that the necessary condition (3.1) is satisfied, we obtain an expression for the AM of a system having a linear integral, as a sum of two components

$$\mathbf{G} = G_{rc}\mathbf{r}_c^0 + A_2\mathbf{x}_c^a = G_{rc}\mathbf{r}_c^0 + A_2\mathbf{r}_c^0 \times (\mathbf{r}_c^0)^{\cdot}_{E_1} \tag{4.7}$$

Let $\mathbf{v}_c^a$ be the absolutely velocity of the centre of inertia C. Then formula (4.7) becomes

$$\mathbf{G} = G_{rc}\mathbf{r}_c^0 + A_2|\mathbf{r}_c|^{-2}\,\mathbf{r}_c \times \mathbf{v}_c^a \tag{4.8}$$

We have thus shown that if a linear integral exists, the AM $\mathbf{G}$ of the system has two components, one collinear with $\mathbf{r}_c$ and given by (3.3) or (4.2), the other equal to the AM of a point mass of mass $A_2|\mathbf{r}_c|^{-2}$ placed at the centre of inertia of the system. The second component is also equal to the AM of a point mass with radius vector $(M\mathbf{r}_c^2A_2^{-1})^{-1/2}\mathbf{r}_c$ and mass equal to the mass of the system M.

5. We will now derive conditions for the existence of a quadratic integral of system (1.3) in the form

$$\tfrac{1}{2}\mathbf{G} \cdot (B\mathbf{G}) + \mathbf{m} \cdot \mathbf{G} + \mathbf{n} \cdot \mathbf{s} + \varphi(t) = \text{const} \tag{5.1}$$

assuming that $J, B, \mathbf{m}, \mathbf{n}, \mathbf{G}_i, \varphi \in C^1[0, +\infty)$. Differentiating (5.1), by virtue of system (1.3) we obtain the identity

$$(B\mathbf{G}+\mathbf{m})\cdot(\mathbf{G}\times\mathbf{x}_{21}+\Lambda\mathbf{x}_{21}+\mathbf{s}\times\mathbf{a}+\mathbf{L})+\tfrac{1}{2}\mathbf{G}\cdot(\dot{B}\mathbf{G})+\dot{\mathbf{m}}\cdot\mathbf{G}+\dot{\mathbf{n}}\cdot\mathbf{s}+\mathbf{n}\cdot(\mathbf{s}\times\mathbf{x}_{21})+\dot{\varphi}\equiv 0 \tag{5.2}$$

Putting $\mathbf{x} = \mathbf{x}_{21}$ and substituting $\mathbf{G} = J\mathbf{x} + \mathbf{G}_2$ into (5.2), we obtain homogeneous identities in $\mathbf{x}$ and $\mathbf{s}$

$$(BJ\mathbf{x}) \cdot (J\mathbf{x} \times \mathbf{x}) = 0 \tag{5.3}$$

$$(BJ\mathbf{x})\cdot(\mathbf{G}_2\times\mathbf{x}+\Lambda\mathbf{x})+(B\mathbf{G}_2+\mathbf{m})\cdot(J\mathbf{x}\times\mathbf{x})+\tfrac{1}{2}\mathbf{x}\cdot(J\dot{B}J\mathbf{x})=0 \tag{5.4}$$

$$(BJ\mathbf{x}) \cdot (\mathbf{s} \times \mathbf{a}) + \mathbf{n} \cdot (\mathbf{s} \times \mathbf{x}) = 0 \tag{5.5}$$

$$\mathbf{x}\cdot(JB\mathbf{L})+(B\mathbf{G}_2+\mathbf{m})\cdot(\mathbf{G}_2\times\mathbf{x}+\Lambda\mathbf{x})+\mathbf{x}\cdot(J\dot{B}\mathbf{G}_2)+\dot{\mathbf{m}}\cdot(J\mathbf{x})=0 \tag{5.6}$$

$$(B\mathbf{G}_2+\mathbf{m})\cdot(\mathbf{s}\times\mathbf{a})+\dot{\mathbf{n}}\cdot\mathbf{s}=0 \tag{5.7}$$

$$(B\mathbf{G}_2+\mathbf{m})\cdot\mathbf{L}+\tfrac{1}{2}\mathbf{G}_2\cdot(\dot{B}\mathbf{G}_2)+\dot{\mathbf{m}}\cdot\mathbf{G}_2+\dot{\varphi}=0 \tag{5.8}$$

From now on we shall assume that the A_i are all different.

Proposition 1. The operator B in (5.1) has the form

$$B = \nu_1(t)J^{-1} + \nu_2(t)E \tag{5.9}$$

Indeed, identity (5.3) is equivalent to the statement that the vectors $BJ\mathbf{x}, J\mathbf{x}, \mathbf{x}$ are coplanar, whatever $\mathbf{x}$. If $\mathbf{x}$ is not collinear with a principal axis of inertia, the vectors $J\mathbf{x}$ and $\mathbf{x}$ are not collinear and then $BJ\mathbf{x} = \nu_1\mathbf{x} + \nu_2 J\mathbf{x}$, which implies (5.9).

Proposition 2. A necessary condition for the existence of an integral is that B and $\mathbf{n}$ must be expressible as $B = \nu J^{-1}$, $\mathbf{n} = \nu\mathbf{a}$.

Identity (5.5) holds for all $\mathbf{s}$ only if $\mathbf{n}\times\mathbf{x} \equiv \mathbf{a}\times BJ\mathbf{x}$. Putting $\mathbf{x} = \mathbf{e}_i$, consider the scalar product of the last identity and $\mathbf{e}_j$. By (5.9), we obtain $\mathbf{e}_k \cdot (\mathbf{n} - (\nu_1 + \nu_2 A_i)\mathbf{a}) = 0$ for $i \neq k, i, k = 1, 2, 3$. This is a system of linear homogeneous equations for the components of $\mathbf{n}$, and since $\mathbf{a} \neq 0$ it has a non-trivial solution. Hence its determinant must vanish, $\nu_2^3(A_1-A_2)(A_2-A_3)(A_1-A_3) = 0$. Since the A_i are all different, it follows that $\nu_2 \equiv 0$. It now follows from the system that $\mathbf{n} = \nu_1\mathbf{a}$ and it follows from (5.9) that $B = \nu_1 J^{-1}$.

Proposition 3. The parameter $\mathbf{n}(t)$ in the integral is a unit vector in the direction of $\mathbf{r}_c(t)$ and the operator B may be written as $B = a^{-1}J^{-1}$.

Identity (5.7) holds only if

$$\dot{\mathbf{n}}+\mathbf{a}\times(B\mathbf{G}_2+\mathbf{m})=0 \tag{5.10}$$

By Proposition 2, $\mathbf{n} \parallel \mathbf{a}$, and it then follows from (5.10) that $|\, \mathbf{n} \,| = \text{const}$, so that $|\, \nu\mathbf{a} \,| = \text{const}$. Since the left-hand side of (5.1) is defined, apart from a constant factor, we can put $|\, \nu\mathbf{a} \,| = 1$ and $\nu = a^{-1}$, Proposition 3 follows from Proposition 2.

Proposition 4. A necessary condition for the existence of an integral is that in a basis E_4 rotating relative to E_2 at angular velocity

$$\mathbf{x}_{42} = J^{-1}\mathbf{G}_2 + a\mathbf{m} \tag{5.11}$$

the direction of the radius vector of the centre of inertia is fixed.

Taking into account that $\mathbf{n} = \mathbf{a}^0, B = a^{-1}J^{-1}$, we can write condition (5.10) as $\dot{\mathbf{a}}^0 + \mathbf{a}^0\times\mathbf{x}_{42} = 0$, whence it follows that

$$(\mathbf{a}^0)^{\cdot}_{E_4} = 0 \tag{5.12}$$

Proposition 5. A necessary condition for the existence of an integral is that

$$(\mathbf{a}J)^{\cdot}_{E_4} = 2a\Lambda \tag{5.13}$$

Identity (5.4) has the form $\mathbf{x} \cdot (F\mathbf{x}) \equiv 0$, where $F = \Lambda + M(\mathbf{x}_{42})J - (2a)^{-1}(aJ)$, and it will hold only if F is a skew-symmetric operator. Since $F + F^T = 0$, it follows that $a^{-1}(aJ)^{\cdot} + JM(\mathbf{x}_{42}) - M(\mathbf{x}_{42})J = 2\Lambda$, which implies (5.13) if we take into account the truth of (5.11) and use the following relationship (see, for example, [2, p. 145])

$$(\mathbf{a}J)^{\cdot}_{E_2} = (\mathbf{a}J)^{\cdot}_{E_4} + M(\mathbf{x}_{42})aJ - aJM(\mathbf{x}_{42})$$

Proposition 6. A necessary condition for the existence of an integral is that

$$\mathbf{L} = \Lambda\mathbf{x}_{42} + (\mathbf{G})^{\cdot}_{E_4} \tag{5.14}$$

Since $B\mathbf{G}_2 + \mathbf{m} = a^{-1}\mathbf{x}_{42}$, identity (5.6) is equivalent to the condition $\mathbf{L} = \mathbf{G}_2 \times \mathbf{x}_{42} - \Lambda\mathbf{x}_{42} - aJ\dot{B}\mathbf{G}_2 - aJ\mathbf{m}^{\cdot}$. Since by (1.2) we have $\mathbf{G}_2 = J\mathbf{x}_{42} + \mathbf{G}_4$, it follows from (5.11) that $\mathbf{m} = -a^{-1}J^{-1}\mathbf{G}_4$, and, noting that $B = a^{-1}J^{-1}$, we can express $\mathbf{L}$ as

$$\mathbf{L} = (J\mathbf{x}_{42} + \mathbf{G}_4) \times \mathbf{x}_{42} - \Lambda\mathbf{x}_{42} + (aJ)^{\cdot} a^{-1}\mathbf{x}_{42} + \dot{\mathbf{G}}_4$$

Changing to differentiation in E_4 and using (5.13), we obtain (5.14).

Proposition 7. The parameter $\varphi(t)$ in the integral has the form

$$\varphi(t) = \tfrac{1}{2}\mathbf{m} \cdot (B^{-1}\mathbf{m}) = \tfrac{1}{2}a^{-1}\mathbf{G}_4 \cdot (J^{-1}\mathbf{G}_4) \tag{5.15}$$

To prove this, we set in $\mathbf{G} = -B^{-1}\mathbf{m}$ in identity (5.2). Since the sum of terms involving $\mathbf{s}$ vanishes identically, we obtain $2\dot{\varphi} = -\mathbf{m} \cdot (B^{-1}\dot{B}B^{-1}\mathbf{m}) + 2\dot{\mathbf{m}}(B^{-1}\mathbf{m})$, which implies the first equality in (5.15). Since $\mathbf{m} = -(aJ)^{-1}\mathbf{G}_4$, the proof is complete.

Proposition 8. The quadratic integral (5.1) exists if and only if conditions (5.12)–(5.14) are satisfied, in which case $B = (aJ^{-1}$, $\mathbf{m} = -(aJ)^{-1}\mathbf{G}_4$, $\mathbf{n} = \mathbf{a}^0$, $\varphi = (2a)^{-1}\mathbf{G}_4 \cdot (J^{-1}\mathbf{G}_4)$.

We have thus found explicit expressions for the parameters in the integral and obtained criterial conditions for its existence.

6. We now clarify the mechanical meaning of the quadratic integral. By Proposition 8, we can express the integral (5.1) as

$$a^{-1}(\mathbf{G} - \mathbf{G}_4) \cdot (J^{-1}(\mathbf{G} - \mathbf{G}_4)) + 2\mathbf{a}^0 \cdot \mathbf{s} = \text{const} \tag{6.1}$$

Since $\mathbf{G} = \mathbf{G}_1$, it follows from (1.2) that $\mathbf{G} - \mathbf{G}_4 = J\mathbf{x}_{41}$, and we have the following equivalent form of the integral

$$a^{-1}\mathbf{x}_{41} \cdot (J\mathbf{x}_{41}) + 2\mathbf{a}^0 \cdot \mathbf{s} = \text{const} \tag{6.2}$$

where $\mathbf{x}_{41} \cdot (J\mathbf{x}_{41})/2 = T_4^e$ is the kinetic energy of the system in translational motion together with E_4 (i.e. in particular, in rotation together with $\mathbf{r}_c$). Since $a\mathbf{a}^0 \cdot \mathbf{s} = \mathbf{a} \cdot \mathbf{s} = P\mathbf{r}_c \cdot \mathbf{s} = V$ is the potential energy of the system, we can express the integral (6.2) as

$$T_4^e + V = P(t)|\mathbf{r}_c(t)|\text{const} \tag{6.3}$$

Thus, if the compound mechanical system of variable configuration and mass that we have been studying is such that its centre of inertia is not the fixed point of the carrier and its principal moments of inertia are pairwise distinct, we obtain the following result.

Proposition 9. If the system has an integral that is quadratic in the components of the angular momentum, it may be written in one of the forms (6.1)–(6.3).

If the quadratic integral exists, it expresses the fact that the sum of the kinetic energy of the system in translational motion together with the basis E_4 and the potential energy of the system in a uniform gravitational field is directly proportional to the absolute value of the static moment of the system about the fixed point. In the special case when $P(t)|\mathbf{r}_c(t)|$ = const, the integral takes the form of the physical energy integral

$$T_4^e + V = \text{const} \tag{6.4}$$

It will be shown below that condition (5.13) determines the angular velocity $\mathbf{x}_{43}$ of the basis E_4 relative to the principal basis E_3, and consequently one can also determine the angular velocity $\mathbf{x}_{42}$ of E_4 relative to the carrier. Thus, all the parameters in the integral and the criterial conditions may be expressed explicitly in terms of given quantities.

7. Let us change to differentiation in E_3 in condition (5.13)

$$(aJ)^{\cdot}_{E_3} = 2a\Lambda + M(\mathbf{x}_{43})aJ - aJM(\mathbf{x}_{43}) \tag{7.1}$$

Put $\lambda_{ij} = \mathbf{e}_i \cdot (\Lambda \mathbf{e}_j)$. It follows from the definition (1.5) that

$$\lambda_{ij} = \lambda_{ji} = \sum \dot{m}_n (r_n^2 \delta_{ij} - r_n^{(i)} r_n^{(j)})$$

Multiplying (7.1) on the left by $\mathbf{e}_i$ and on the right by $\mathbf{e}_j$, we obtain

$$(aA_i)^{\cdot}\delta_{ij} = 2a\lambda_{ij} + a(A_i - A_j)(\mathbf{e}_i \times \mathbf{e}_j)\cdot\mathbf{x}_{43}, \quad i,j = 1,2,3 \tag{7.2}$$

When $i = j$ this implies conditions linking the given functions

$$(\ln a)^{\cdot} = 2\lambda_{11}A_1^{-1} - (\ln A_1)^{\cdot} = 2\lambda_{22}A_2^{-1} - (\ln A_2)^{\cdot} = 2\lambda_{33}A_3^{-1} - (\ln A_3)^{\cdot} \tag{7.3}$$

When $i \neq j$ conditions (7.2) give the angular velocity components

$$x_{43}^{(k)} = -2\lambda_{ij}(A_k\alpha_k)^{-1}, \quad k \neq i, \; k \neq j \tag{7.4}$$

Thus, condition (5.13) is equivalent to the three scalar conditions (7.3) and to specifying the angular velocity $\mathbf{x}_{43}$ in the form (7.4).

It follows from this relationship between $\mathbf{x}_{43}$ and λ_{ij} that the basis E_4 will coincide with the principal basis E_3 if and only if a basis made up of eigenvectors of the operator Λ coincides with E_3.

8. Assuming that the criterial conditions (5.12)–(5.14) are satisfied, we will now transform the initial system (1.3) and directly verify that these conditions are sufficient for the existence of a quadratic integral. To that end we change in (1.3) to differentiation in the basis E_4 and substitute $\mathbf{L}$ from (5.14)

$$(\mathbf{G})^{\cdot}_{E_4} = \mathbf{G}\times\mathbf{x}_{41} + \Lambda\mathbf{x}_{41} + (\mathbf{G}_4)^{\cdot}_{E_4} + \mathbf{s}\times\mathbf{a}, \quad (\mathbf{s})^{\cdot}_{E_4} = \mathbf{s}\times\mathbf{x}_{41}$$

Substituting $\mathbf{G} = J\mathbf{x}_{41} + \mathbf{G}_4$ and Λ from (5.13) into this equality, we obtain the system

$$(2a)^{-1}(aJ)^{\cdot}_{E_4}\mathbf{x}_{41} + aJ(a^{-1}\mathbf{x}_{41})^{\cdot}_{E_4} = (J\mathbf{x}_{41} + \mathbf{G}_4)\times\mathbf{x}_{41} + \mathbf{s}\times\mathbf{a}, \quad (\mathbf{s})^{\cdot}_{E_4} = \mathbf{s}\times\mathbf{x}_{41} \tag{8.1}$$

Taking the scalar product of the first equation in (8.1) and the vector $a^{-1}\mathbf{x}_{41}$, with due attention to condition (5.12) and the second equation in (8.1), we obtain

$$(a^{-1}\mathbf{x}_{41}\cdot(aJa^{-1}\mathbf{x}_{41}))^{\cdot} = 2\mathbf{x}_{41}\cdot(\mathbf{s}\times\mathbf{a}^0) = -2\mathbf{a}^0\cdot(\mathbf{s})^{\cdot}_{E_4} = -2(\mathbf{a}^0\cdot\mathbf{s})^{\cdot}$$

which implies the existence of the integral (6.2).

We will use one more transformation. Denote $\Phi = (aJ)^{1/2}$. Introducing the variables

$$\mathbf{y} = a^{-1}\Phi\mathbf{x}_{41}, \quad \tau = \int a(t)dt$$

we can rewrite system (8.1) in the form

$$\left.\frac{d\mathbf{y}}{d\tau}\right|_{E_4} = (a^3A_1A_2A_3)^{-1/2}(\Phi^2\mathbf{y} + \Phi\mathbf{G}_4)\times\mathbf{y} + \mathbf{c}\times\mathbf{y} + \Phi^{-1}(\mathbf{s}\times\mathbf{a}^0), \quad \left.\frac{d\mathbf{s}}{d\tau}\right|_{E_4} = \mathbf{s}\times\Phi^{-1}y \tag{8.2}$$

If the direction of $\mathbf{a}$ is constant in E_4, multiplication of (8.2) by $\mathbf{y}$ will yield an integral equivalent to (6.2)

$$y^2 + 2\mathbf{a}^0 \cdot \mathbf{s} = \text{const.}$$

The vector $\mathbf{c}$ in Eq. (8.2) is defined by the condition $2aM(\mathbf{c}) = \Phi M(\mathbf{x}_{43})\Phi^{-1} + \Phi^{-1}M(\mathbf{x}_{43})\Phi - 2M(\mathbf{x}_{43})$, whence it follows that the components of $\mathbf{c}$ are

$$c^{(k)} = (\sqrt{A_i} - \sqrt{A_j})^2 (2a\sqrt{A_i A_j})^{-1} x_{43}^{(k)}, \quad k \neq i, \ k \neq j \tag{8.3}$$

9. The results for model M may be obtained for $\Lambda \equiv 0$. It then follows from (7.4) that $\mathbf{x}_{43} = 0$ and the bases E_4 and E_3 coincide. The integral (6.1) becomes

$$a^{-1}(\mathbf{G} - \mathbf{G}_3) \cdot (J^{-1}(\mathbf{G} - \mathbf{G}_3)) + 2\mathbf{a}^0 \cdot \mathbf{s} = \text{const} \tag{9.1}$$

where $\mathbf{G} - \mathbf{G}_3 = J\mathbf{x}_{31}$, and this integral may also be written in the form of (6.2): $a^{-1}\mathbf{x}_{31} \cdot (J\mathbf{x}_{31}) + 2\mathbf{a}^0 \cdot \mathbf{s} = \text{const}$.

The necessary and sufficient conditions for the existence of a quadratic integral are obtained from (5.12), (5.14) and (7.3)

$$A_i = a^{-1}c_i \ \ (c_i = \text{const}), \quad \mathbf{r}_c^0|_{E_3} = \text{const}, \quad \mathbf{L} = \sum \dot{m}_n \mathbf{r}_n \times \mathbf{u}_n = (\mathbf{G}_3)^{\cdot}_{E_3} \tag{9.2}$$

Makayev (see the publications cited in the footnote to the first page of this paper) has investigated the conditions of existence of an integral of the form

$$\mathbf{G} \cdot (B\mathbf{G}) + \mathbf{m} \cdot \mathbf{G} + \mathbf{n} \cdot \mathbf{s} = \text{const}$$

This integral is the special case of (5.1) with $\varphi = \text{const}$. Since, by (5.13), the operator aJ is constant in E_3, it follows from (5.15) that the angular momentum G_3 is constant in E_3 and, by (9.2), $\mathbf{L} = 0$. Thus, a quadratic integral in Makeyev's sense may be written in the form of (9.1), and the conditions for its existence are

$$A_i = a^{-1}c_i, \ (\mathbf{r}_c^0)^{\cdot}_{E_3} = 0, \ (\mathbf{G}_3)^{\cdot}_{E_3} = 0, \ \mathbf{L} = 0.$$

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